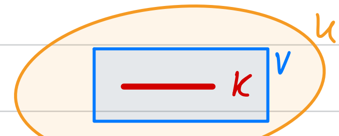
Math 565: Functional Analysis
Lecture 8

Locally compact Hausdorff facts. A Hausdorff top. space X is called *locally compact* if every point admits a compact neighbourhood. The following are key facts we will use for a lcH X. Let K be compact and $U \supseteq K$ be open.



(a) There is open $V \subseteq X$ s.t. $K \subseteq V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact.

(b) Urysohn for lcH: $\exists f \in C_c(X)$ with $0 \leq f \leq 1$ s.t. $f|_K \equiv 1$ and $f|_{U^c} = 0$.

(c) Partition of unity: If $\{U_i\}_{i<n}$ is an open cover of K then $\exists \{f_i\}_{i<n} \subseteq C_c(X, [0,1])$ with $\sum_{i<n} f_i|_K = 1$ and $\operatorname{supp} f_i \subseteq U_i$ for each $i<n$. This family $\{f_i\}_{i<n}$ is called a *partition of unity* (i.e. the constant 1 function) *on K subordinate to* $\{U_i\}_{i<n}$.

We now try to capture the closure of $C_c(X)$ inside $BC(X)$ with uniform norm. Let

$$C_0(X) := \{f \in BC(X) : f \text{ vanishes outside compact sets}\},$$

where we say that $f \in C(X)$ *vanishes outside compact sets* (or *vanishes at ∞*) if $\forall \varepsilon > 0$ the set $\{|f| \geq \varepsilon\}$ is compact.

Remark on the name. For lcH spaces, $C_0(X)$ is exactly the set of functions $f \in C(X)$ which admit an extension to a continuous function $\bar{f}$ on the one-point compactification $\bar{X} := X \cup \{\infty\}$ such that $\bar{f}(\infty) = 0$, hence the name *vanishing at ∞*.

Prop. For any top space X, $\overline{C_c(X)} \subseteq C_0(X)$; and if X is lcH, then $\overline{C_c(X)} = C_0(X)$.

Proof. Let $f_n \in C_c(X)$ and $f_n \to_u f$, so $f \in BC(X)$. Let $\varepsilon > 0$, so there is $n \in \mathbb{N}$ with $\|f_n - f\|_u < \varepsilon$, in particular, $|f|_{K^c}| < \varepsilon$ where $K := \operatorname{supp} f_n$, so $\{|f| \geq \varepsilon\} \subseteq K$, hence $f \in C_0(X)$.

Now if X is lcH and $f \in C_0(X)$, take $\varepsilon > 0$. Then $K := \{f \geq \frac{\varepsilon}{2}\}$ is compact, so by Fact (b), there is $g \in C_c(X)$ with $\mathbb{1}_K \leq g \leq 1$, in particular, $g|_K \equiv 1$. Then $f \cdot g \in C_c(X)$ and $(f \cdot g)|_K = f|_K$ so $\|f - f \cdot g\|_u \leq \|f|_{K^c}\| + \|(f \cdot g)|_{K^c}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\boxed{\text{QED}}$

Corollary. $C_c(X)^* \cong C_0(X)^*$, i.e. they are isometrically isometry.

Proof. More general statement is in HW. $\square$

Def. Let X be a top. space. Call an (unsigned) Borel measure μ on X a *Radon measure* if it is *finite on compact sets*, *outer regular* on all Borel $B \subseteq X$, i.e.

$$\mu(B) = \inf\{\mu(U) : U \supseteq B \text{ open}\},$$

and *tight* on open sets $U \subseteq X$, i.e.

$$\mu(U) = \sup\{\mu(K) : K \subseteq U \text{ compact}\}.$$

Examples. Haar measures on lcH groups are Radon measures. In particular, the Lebesgue measure on $\mathbb{R}^d$ and Bernoulli$(\frac{1}{2})$ measure on $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}} = 2^{\mathbb{N}}$ are Radon.

Theorem. If X is 2^{nd} ctbl and either Polish (2^{nd} ctbl, admitting a complete metric) or lcH then all finite Borel measures on X are Radon.

Proof. We proved for Polish last semester and for lcH is HW. $\square$

Facts. (a) Regularity, tightness (hence being Radon) are measure equivalence invariants for finite Borel measures on X, i.e. if μ, ν are finite Borel measures on X and $\mu \sim \nu$, then

$$\mu \text{ is regular/tight/Radon iff } \nu \text{ is. (HW)}$$

(b) If X is lcH and μ is a Radon measure on X then $C_c(X) \subseteq L^p(X,\mu)$ is dense in the L^p norm, for all $1 \le p < \infty$. (HW)

Def. A (finite) complex Borel measure ρ on a top. space X is called *Radon* if $|\rho|$ is Radon.

Prop. ρ is Radon (i.e. $|\rho|$ is Radon) $\iff$ ρ_i is Radon for all $0 \le i < 4$, where $\rho = (\rho_0 - \rho_1) + i(\rho_2 - \rho_3)$ s.t. $\rho_0 \perp \rho_1$, $\rho_2 \perp \rho_3$ are unsigned measures.

Proof. Let $\mu := \rho_0 + \rho_1 + \rho_2 + \rho_3$. It is enough to show that $|\rho| \sim \mu$ because then $|\rho|$ Radon

$\Leftrightarrow$ μ Radon $\Leftrightarrow$ $\forall i$, ρ_i Radon. For $|\rho| \sim \mu$, it is clear that $|\rho| \ll \mu$ since
(*) $|\rho|(B) = \int_B \left|\frac{d\rho}{d\mu}\right| d\mu$, so suffices to show that $|\rho| \gg |\mathrm{Re}\,\rho| = \rho_0 + \rho_1$ and $|\rho| \gg |\mathrm{Im}\,\rho| = \rho_2 + \rho_3$. We abbreviate (*) by $d|\rho| = \left|\frac{d\rho}{d\mu}\right| d\mu$. Now observe:

$$d|\rho| = \left|\frac{d\rho}{d\mu}\right| d\mu \geq \left|\mathrm{Re}\frac{d\rho}{d\mu}\right| d\mu = \left|\frac{d\,\mathrm{Re}\,\rho}{d\mu}\right| d\mu = \left|\frac{d\rho_0}{d\mu} - \frac{d\rho_1}{d\mu}\right| d\mu \underset{\rho_0 \perp \rho_1}{=} \left(\frac{d\rho_0}{\cancel{d\mu}} + \frac{d\rho_1}{\cancel{d\mu}}\right)\cancel{d\mu} = d\rho_0 + d\rho_1,$$

so $\rho \gg \rho_0 + \rho_1$. Similarly, $\rho \gg \rho_2 + \rho_3$. $\boxed{\text{QED}}$

Let $RM_{\mathbb{C}}(X)$ denote the space of all (finite) complex Radon measures on a top. space X.

Fact. $RM_{\mathbb{C}}(X) \subseteq M_{\mathbb{C}}(X, \mathcal{B}(X))$ (Borel sets of X) is a closed subspace of the Banach space $M_{\mathbb{C}}(X, \mathcal{B}(X))$, hence a Banach space itself with the total variation norm $\|\rho\|_{TV}$.

Proof. Easy to prove directly using the absolutely convergent series criterion, but also follows from the following theorem. $\square$

Reisz representation theorem. For an lcH space X, $C_0(X, \mathbb{C})^* \cong RM_{\mathbb{C}}(X)$, more precisely, the map $\rho \mapsto I_\rho : RM_{\mathbb{C}}(X) \to C_0(X, \mathbb{C})^*$ is an isometric isomorphism.

We will prove this for the rest of this section.

Proof of isometry. Fix $\rho \in RM_{\mathbb{C}}(X)$. Recall that $\|\rho\|_{TV} := |\rho|(X) = \sup_{|\varphi| \leq 1} \left|\int \varphi \, d\rho\right|$ (achieved by $\varphi := \overline{\mathrm{sgn}\frac{d\rho}{d\mu}}$), where φ ranges over Borel functions. On the other hand,
$\|I_\rho\| = \sup_{|f| \leq 1} |I_\rho(f)| = \sup_{|f| \leq 1} \left|\int f \, d\rho\right|$, where f ranges over $C_0(X)$, so $\|I_\rho\| \leq \|\rho\|_{TV}$. To show equality, fix a Borel $|\varphi| \leq 1$ with $\left|\int \varphi \, d\rho\right| \approx_{\varepsilon/2} \|\rho\|_{TV}$.
But $\varphi \in L^1(X, |\rho|)$ since $|\varphi| \leq 1$ and $|\rho|$ is finite, hence $\exists f \in C_c(X)$ with $\|\varphi - f\|_1 < \frac{\varepsilon}{2}$, because $C_c(X)$ is dense in $L^1(X, |\rho|)$. Then $\left|\int f \, d\rho - \int \varphi \, d\rho\right| \leq \int |f - \varphi| \, d|\rho| < \frac{\varepsilon}{2}$, hence $\left|\int f \, d\rho\right| \approx_{\varepsilon/2} \left|\int \varphi \, d\rho\right| \approx_{\varepsilon/2} \|\rho\|_{TV}$, so $\|I_\rho\| = \|\rho\|_{TV}$. $\boxed{\text{QED}}$

It "remains" to prove the surjectivity of $\rho \mapsto I_\rho$. Fix $I \in C_0(X)^*$. Note that for all $f \in C_0(X)$,

$$I(f) = I(\mathrm{Re} f + i \,\mathrm{Im} f) = I(\mathrm{Re} f) + i\, I(\mathrm{Im} f),$$

so I is determined by its values on $C_0(X, \mathbb{R})$, i.e. $C_0(X, \mathbb{C}) \cong_{\mathbb{R}} \overset{1}{C_0(X,\mathbb{R})} \times \overset{i}{C_0(X,\mathbb{R})}$. So let $J := I|_{C_0(X,\mathbb{R})}$ and $J = \mathrm{Re} J + i \,\mathrm{Im} J$, where $\mathrm{Re} J$ and $\mathrm{Im} J$ are real-linear bdd functionals on $C_0(X, \mathbb{R})$. This reduces to proving

<u>Real Reisz representation theorem.</u> For an lcH space X, $C_0(X,\mathbb{R})^* \cong RM_{\mathbb{R}}(X)$, more precisely, the map $\rho \mapsto I_\rho : RM_{\mathbb{R}}(X) \to C_0(X, \mathbb{R})^*$ is an isometric isometry.

Indeed, we would then apply this to $\mathrm{Re} J$ and $\mathrm{Im} J$, obtaining $\rho_r, \rho_i \in RM_{\mathbb{R}}(X)$ such that $\mathrm{Re} J = I_{\rho_r}$ and $\mathrm{Im} J = I_{\rho_i}$. Then $\rho := \rho_r + i \rho_i$ is as desired: note that it is Radon by the Proposition above since ρ_r and ρ_i are Radon. Finally, $J = I_{\rho_r} + i I_{\rho_i} = I_\rho$, so $I = I_\rho$ by the uniqueness of linear extension of J to $C_0(X, \mathbb{C})$.